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Abstract

This paper examines optimal portfolio selection using quantile-based risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR). We address the case of a singular covariance matrix of asset returns, which leads to an optimization problem with infinitely many solutions. An analytical form for a general solution is derived, along with a unique solution that minimizes the L_2-norm. We also show that the general solution reduces to the standard optimal portfolio for VaR and CVaR when the covariance matrix is non-singular.

Keywords: Minimum VaR portfolio, Minimum CVaR portfolio, Singular covariance matrix, Linear ill-posed problems

1 Introduction

Modern portfolio theory offers an intelligent approach to making investment decisions based on mathematical concepts. In the pioneering work of [15], the basic concepts were introduced through the mean-variance framework. In this framework, the investor allocates funds based on the trade-off between the portfolio's return and risk. The optimal portfolio is selected by maximizing the expected portfolio return subject to achieving a specified level of portfolio risk or, equivalently, by minimizing portfolio risk subject to achieving a specified level of expected portfolio return. When assessing portfolio risk, various measures can be considered. The most basic portfolio risk measure is variance, however, this risk measure is often deemed inappropriate as it considers two-sided risk [14]. Recently, quantile-based risk measures, such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), have gained popularity and are, for instance, recommended by the Basel Committee on Banking Supervision [1, 2]. Therefore, this paper focuses on optimal portfolios obtained by minimizing VaR and CVaR.

Investment decisions and the composition of the investment portfolio depends on the expected returns and risk of different assets and, crucially, the correlations between the returns of the assets. These, in turn, are estimated from financial market data and used as input when investment strategies are formulated and investment decisions taken. The most common estimators for the covariance matrix are the sample estimator and the maximum likelihood estimator (see, e.g., [3, 4, 12, 13, 16]). Both estimators are positive definite when there are more observations in the data than the number of assets in the investment portfolio. However, having fewer observations than the size of the portfolio leads to singular estimators of the covariance matrix. Another source of singularity for the covariance matrix can arise from multicollinearity, especially in investment portfolios with a large number of assets. This is where our contribution comes in.

We contribute to the existing literature by considering minimum VaR and CVaR optimal portfolios when the covariance matrix of asset returns is singular. Since the covariance matrix is singular, the optimization problem has an infinite number of solutions. Therefore, following a similar philosophy as in [10, 11, 17], we present both a general solution and a unique solution with the smallest L_2-norm. It is worth noting that the solution with the smallest L_2-norm has gained attention in the statistical analysis of optimal portfolio weights and related quantities (see, e.g., [3, 5–9]).

The rest of the paper is organized as follows. Section 2 introduces the concept of minimum VaR and CVaR optimal portfolios. After that, Section 3 presents the main contribution of the paper, focusing on the case of a singular covariance matrix.

2 Minimum VaR and CVAR optimal portfolios

Let $\mathbf{x}_t$ be a p-dimensional vector of asset returns at time point t, $t = 1, \ldots, N$, following a p-dimensional Gaussian distribution with mean vector $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, i.e. $\mathbf{x}_t \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let also the second moment of $\mathbf{x}_t$ be finite, and assume $rank(\boldsymbol{\Sigma}) = r \leq p$, meaning that the covariance matrix $\boldsymbol{\Sigma}$ may be singular. Furthermore, let $\mathbf{w}$ be a p-dimensional vector of portfolio weights such that $\mathbf{1}^\top \mathbf{w} = 1$, where $\mathbf{1}$ denotes the p-dimensional vector of ones.

[1,2] used Value-at-Risk (VaR) as a risk measure to find the optimal portfolio weights $\mathbf{w}$ through the following optimization problem

$$VaR_\alpha \to \min \quad \text{s.t.} \quad \mathbf{1}^\top \mathbf{w} = 1, \tag{1}$$

where the VaR at the confidence level $\alpha \in (1/2, 1)$ is defined as follows

$$\mathbb{P}\left(X_{\mathbf{w}} < -VaR_\alpha\right) = 1 - \alpha \tag{2}$$

with $X_{\mathbf{w}} = \mathbf{x}^\top \mathbf{w}$. Assuming that $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, it holds that

$$VaR_\alpha(X_{\mathbf{w}}) = -\mathbf{w}^\top \boldsymbol{\mu} - z_{1-\alpha}\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}} \tag{3}$$

with $z_\beta = \Phi^{-1}(\beta)$ which corresponds to β-quantile of the standard Gaussian distribution. Therefore, if $\boldsymbol{\Sigma}$ is positive definite, the solution to (1) is expressed as follows

$$\mathbf{w}_{VaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{z_{1-\alpha}^2 - s}} \mathbf{R}\boldsymbol{\mu} \tag{4}$$

with

$$\mathbf{w}_{GMV} = \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}, \qquad R_{GMV} = \frac{\boldsymbol{\mu}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}, \qquad V_{GMV} = \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}}, \tag{5}$$

where $\mathbf{w}_{GMV}$, R_{GMV}, and V_{GMV} are the weights, expected return and variance of the global minimum variance (GMV) portfolio, respectively, and

$$s = \boldsymbol{\mu}^\top \mathbf{R} \boldsymbol{\mu} \qquad \text{with} \qquad \mathbf{R} = \boldsymbol{\Sigma}^{-1} - \frac{\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1}} \tag{6}$$

is a slope parameter of the efficient frontier.

Following [1,2], another popular measure of risk is the Conditional Value-at-Risk (CVaR), which is defined at the confidence level $\alpha \in (1/2, 1)$ as

$$CVaR_\alpha = \mathbb{E}\left(-X_{\mathbf{w}} | - X_{\mathbf{w}} > VaR_\alpha\right). \tag{7}$$

Considering $\mathbf{x} \sim \mathcal{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, we get that

$$CVaR_\alpha(X_{\mathbf{w}}) = -\mathbf{w}^\top \boldsymbol{\mu} - \kappa_{1-\alpha}\sqrt{\mathbf{w}^\top \boldsymbol{\Sigma} \mathbf{w}}, \tag{8}$$

with

$$\kappa_\beta = \frac{-\int_{-\infty}^{-z_\beta} x\phi(x)dx}{1-\beta} = \frac{\exp\left(-z_\beta^2/2\right)}{\sqrt{2\pi}(1-\beta)}, \tag{9}$$

where $\phi(\cdot)$ stands for the density function of the standard Gaussian distribution. Hence, the minimum CVaR optimal portfolio weights are obtained as a solution to the following optimization problem

$$CVaR_\alpha \to \min \quad \text{s.t.} \quad \mathbf{1}^\top \mathbf{w} = 1 \tag{10}$$

and, in the case of the positive definite $\mathbf{\Sigma}$, are expressed as follows

$$\mathbf{w}_{CVaR} = \mathbf{w}_{GMV} + \frac{\sqrt{V_{GMV}}}{\sqrt{\kappa_{1-\alpha}^2 - s}} \mathbf{R}\boldsymbol{\mu}. \tag{11}$$

The optimal portfolio weights $\mathbf{w}_{VaR}$ and $\mathbf{w}_{CVaR}$ in (4) and (11), respectively, are obtained under the assumption of a positive definite covariance matrix $\mathbf{\Sigma}$. In what follows, we consider the case of the singular covariance matrix $\mathbf{\Sigma}$ and, in this context, deliver the corresponding expressions for $\mathbf{w}_{VaR}$ and $\mathbf{w}_{CVaR}$.

3 Main results

Assuming Gaussianity, the optimization problems for obtaining both the VaR and CVaR optimal portfolios in (1) and (10), respectively, can be generalized as follows

$$\begin{aligned} \min_{\mathbf{w} \in \mathbb{R}^p} \quad & -\boldsymbol{\mu}^\top \mathbf{w} + \gamma \sqrt{\mathbf{w}^\top \mathbf{\Sigma} \mathbf{w}} \\ \text{s.t.} \quad & \mathbf{1}^\top \mathbf{w} = 1, \end{aligned} \tag{12}$$

where $\gamma \in \mathbb{R}$ is a given constant and $\mathbf{\Sigma}$ has rank $r < p$. Let us note that for $\alpha \in (1/2, 1)$ we have $\gamma = -z_{1-\alpha} > 0$ in the VaR optimization problem, and $\gamma = -\kappa_{1-\alpha} < 0$ in the CVaR case. Through the SVD the matrix $\mathbf{\Sigma}$ can be represented as

$$\mathbf{\Sigma} = \mathbf{U}\mathbf{S}\mathbf{U}^\top, \text{ where } \mathbf{S} = \begin{bmatrix} \mathbf{D} & 0 \\ 0 & 0 \end{bmatrix}, \mathbf{U} = [\mathbf{U}_1,\ \mathbf{U}_2]. \tag{13}$$

Here, $\mathbf{D}$ is $(r \times r)$ diagonal matrix, and $\mathbf{U}_1$ and $\mathbf{U}_2$ are $(p \times r)$ and $p \times (p - r)$ orthonormal matrices. Then

$$\mathbf{\Sigma} = \mathbf{U}_1 \mathbf{D} \mathbf{U}_1^\top \tag{14}$$

and the pseudo-inverse of $\mathbf{\Sigma}$ is given as

$$\mathbf{\Sigma}^\dagger = \mathbf{U}_1 \mathbf{D}^{-1} \mathbf{U}_1^\top. \tag{15}$$

Theorem 1. *A general solution to the optimization problem in* (12) *with the rank deficient matrix* $\mathbf{\Sigma}$*, represented with the help of the SVD in* (13) - (14)*, is given as*

$$\mathbf{w} = \frac{(1 - \mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2)}{\mathbf{1}^\top \mathbf{\Sigma}^\dagger (\boldsymbol{\mu} - \lambda_1 \mathbf{1})} \mathbf{\Sigma}^\dagger (\boldsymbol{\mu} - \lambda_1 \mathbf{1}) + \mathbf{U}_2 \mathbf{y}_2. \tag{16}$$

Here, $\mathbf{y}_2 \in \mathbb{R}^{p-r}$ *is a vector of free variables and* $\lambda_1 \in \mathbb{R}$ *is given as*

$$\lambda_1 = \begin{cases} \lambda_1^+, & \gamma(1 - \mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2) < 0 \\ \lambda_1^-, & \gamma(1 - \mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2) \geq 0 \end{cases}, \tag{17}$$

with

$$\lambda_1^\pm = \frac{b \pm \sqrt{b^2 - ac + a\gamma^2}}{a}, \tag{18}$$

where

$$a = \mathbf{1}^\top \mathbf{\Sigma}^\dagger \mathbf{1}, \quad b = \boldsymbol{\mu}^\top \mathbf{\Sigma}^\dagger \mathbf{1}, \text{ and } c = \boldsymbol{\mu}^\top \mathbf{\Sigma}^\dagger \boldsymbol{\mu}. \tag{19}$$

Proof. To solve the optimization problem in (12), we introduce an auxiliary variable $v = \sqrt{\mathbf{w}^\top \mathbf{\Sigma} \mathbf{w}} \geq 0$. Then the Lagranian function for (12) is given as

$$\mathcal{L} = -\boldsymbol{\mu}^\top \mathbf{w} + \gamma v + \lambda_1 (\mathbf{1}^\top \mathbf{w} - 1) + \lambda_v (\mathbf{w}^\top \mathbf{\Sigma} \mathbf{w} - v^2), \tag{20}$$

where λ_1, λ_v are Lagrange parameters. In order to obtain the minimal solution the KKT-conditions must be satisfied, i.e.,

$$\frac{\partial L}{\partial \mathbf{w}} = -\boldsymbol{\mu} + \lambda_1 \mathbf{1} + 2\lambda_v \boldsymbol{\Sigma}\mathbf{w} = 0 \tag{21}$$

$$\frac{\partial L}{\partial v} = \gamma - 2\lambda_v v = 0 \tag{22}$$

$$\mathbf{1}^\top \mathbf{w} = 1 \tag{23}$$

$$\mathbf{w}^\top \boldsymbol{\Sigma}\mathbf{w} = v^2. \tag{24}$$

Introduce $\mathbf{y} = \mathbf{U}^\top \mathbf{w}$, where $\mathbf{U}$ is given by the SVD decomposition of $\boldsymbol{\Sigma}$ in (14). Consequently,

$$\mathbf{w} = \mathbf{U}\mathbf{y} = \mathbf{U}_1\mathbf{y}_1 + \mathbf{U}_2\mathbf{y}_2, \quad \mathbf{y} = \begin{pmatrix}\mathbf{y}_1 \\ \mathbf{y}_2\end{pmatrix}, \quad \mathbf{y}_1 \in \mathbb{R}^r, \mathbf{y}_2 \in \mathbb{R}^{p-r}. \tag{25}$$

The the KKT-conditions (21)-(24) can then we rewritten in terms of $\mathbf{y}_1$ and $\mathbf{y}_2$ as

$$-\boldsymbol{\mu} + \lambda_1 \mathbf{1} + 2\lambda_v \mathbf{U}_1\mathbf{D}\mathbf{y}_1 = 0 \tag{26}$$

$$\gamma - 2\lambda_v v = 0 \tag{27}$$

$$\mathbf{1}^\top \mathbf{U}_1\mathbf{y}_1 + \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2 = 1 \tag{28}$$

$$\mathbf{y}_1^\top \mathbf{D}\mathbf{y}_1 = v^2. \tag{29}$$

Multiplying (26) from the left with $\mathbf{D}^{-1}\mathbf{U}_1^\top$ yields

$$\mathbf{y}_1 = \frac{1}{2\lambda_v}\mathbf{D}^{-1}\mathbf{U}_1^\top(\boldsymbol{\mu} - \lambda_1\mathbf{1}). \tag{30}$$

We plug this expression into (29) and obtain

$$(\boldsymbol{\mu} - \lambda_1\mathbf{1})^\top \mathbf{U}_1\mathbf{D}^{-1}\mathbf{U}_1^\top(\boldsymbol{\mu} - \lambda_1\mathbf{1}) = 4\lambda_v^2 v^2.$$

We next use (27) and the expression for the pseudo-inverse $\boldsymbol{\Sigma}^\dagger$ in (15) to obtain

$$(\boldsymbol{\mu} - \lambda_1\mathbf{1})^\top \boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1}) = \gamma^2. \tag{31}$$

Solving (31) for λ_1 results in (18). Assuming λ_1 given by (18), we plug (30) into (28) to obtain the expression for λ_v. We have

$$\frac{1}{2\lambda_v} = \frac{(1 - \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2)}{\mathbf{1}^\top \boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1})} = \frac{(1 - \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2)}{\mp\sqrt{b^2 - ac + a\gamma^2}}. \tag{32}$$

Since $v = \gamma/(2\lambda_v) \geq 0$, see (27), then the expression above must be of a certain sign. This can be guaranteed by selecting $\lambda_1 = \lambda_1^+$ if $\gamma(1 - \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2) < 0$ and $\lambda_1 = \lambda_1^-$ if $\gamma(1 - \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2) \geq 0$, see (17).

Combining (32) with the expression for $\mathbf{y}_1$ in (30) yields

$$\mathbf{U}_1\mathbf{y}_1 = \frac{(1 - \mathbf{1}^\top \mathbf{U}_2\mathbf{y}_2)}{\mathbf{1}^\top \boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1})}\boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1}). \tag{33}$$

Finally, we obtain (16) by using (25), which completes the proof. □

Corollary 1. *When $\boldsymbol{\Sigma}$ has full rank, $\mathbf{w}$ in (16) are the same as $\mathbf{w}_{VaR}$ and $\mathbf{w}_{CVaR}$ in (4) and (11), respectively.*

Proof. For the full rank case, (16) in Theorem 1 we have

$$\mathbf{w} = \frac{1}{\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \lambda_1\mathbf{1})}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \lambda_1\mathbf{1}).$$

where a, b, c, and γ are defined in Theorem 1. Using that

$$\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \lambda_1\mathbf{1}^\top \boldsymbol{\Sigma}^{-1}\mathbf{1} = b - a\lambda_1 = b - b + \sqrt{b^2 - ac + a\gamma^2} = \sqrt{b^2 - ac + a\gamma^2}$$

we get

$$\mathbf{w} = \frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \lambda_1\mathbf{1}) = \frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} - \frac{\lambda_1}{\sqrt{b^2 - ac + a\gamma^2}}\mathbf{1})$$

or since

$$\frac{\lambda_1}{\mathbf{1}^\top\boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1})} = \frac{\frac{b\pm\sqrt{b^2-ac+a\gamma^2}}{a}}{\sqrt{b^2 - ac + a\gamma^2}} = -\frac{1}{a} + \frac{b}{a}\frac{1}{\sqrt{b^2 - ac + a\gamma^2}}$$

we have

$$\mathbf{w} = \frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\boldsymbol{\Sigma}^{-1}\mu - \left(\frac{1}{a} - \frac{b}{a}\frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1}. \tag{34}$$

We now show that $\mathbf{w}_{VaR} = \mathbf{w}$ where $\mathbf{w}_{VaR}$ and $\mathbf{w}$ are given in (4) and (34), respectively. From (4)–(6) we get

$$\mathbf{w}_{VaR} = \frac{1}{a}\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{\sqrt{a}}\frac{1}{\sqrt{\gamma^2 - \boldsymbol{\mu}^\top\mathbf{R}\boldsymbol{\mu}}}\left(\boldsymbol{\Sigma}^{-1} - \frac{1}{a}\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}^\top\boldsymbol{\Sigma}^{-1}\right)\boldsymbol{\mu}$$

where $s = \boldsymbol{\mu}^\top\mathbf{R}\boldsymbol{\mu} = c - b^2/a$ giving

$$\mathbf{w}_{VaR} = \frac{1}{a}\boldsymbol{\Sigma}^{-1}\mathbf{1} + \frac{1}{\sqrt{a}}\frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\boldsymbol{\Sigma}^{-1}\left(\mu - \frac{b}{a}\mathbf{1}\right)$$

or rearranging terms

$$\mathbf{w}_{VaR} = \frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\boldsymbol{\Sigma}^{-1}\mu - \left(\frac{1}{a} - \frac{b}{a}\frac{1}{\sqrt{b^2 - ac + a\gamma^2}}\right)\boldsymbol{\Sigma}^{-1}\mathbf{1} = \mathbf{w}.$$

Similarly, we get the expression for the CVaR optimal portfolio weights. The proof of the corollary is now complete. □

As there are infinitely many solutions so there is a possibility to select a solution with some additional properties. In particular, similar to [11], it could be interesting to find a solution that has a minimum L_2-norm.

Theorem 2. *The unique minimum norm solution to* (12) *is given as*

$$\mathbf{w}_{\min} = \frac{1}{\|\mathbf{U}_2^\top\mathbf{1}\|^2\,\|\mathbf{x}\|^2 + 1}\left(\mathbf{x} + \mathbf{U}_2\mathbf{U}_2^\top\mathbf{1}\|\mathbf{x}\|^2\right), \tag{35}$$

with

$$\|\mathbf{w}_{\min}\| = \frac{\|\mathbf{x}\|}{\sqrt{\|\mathbf{U}_2^\top\mathbf{1}\|^2\,\|\mathbf{x}\|^2 + 1}}, \tag{36}$$

where

$$\mathbf{x} = \frac{1}{\mathbf{1}^\top\boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1})}\boldsymbol{\Sigma}^\dagger(\boldsymbol{\mu} - \lambda_1\mathbf{1}) \tag{37}$$

with a, b and c are given in (19) *and $\lambda_1 = \lambda_1^-$ if $\gamma > 0$ and $\lambda_1 = \lambda_1^+$ if $\gamma < 0$, see* (18).

Proof. From (25), we calculate

$$\|\mathbf{w}\|^2 = \|\mathbf{U}_1\mathbf{y}_1\|^2 + \|\mathbf{U}_2\mathbf{y}_2\|^2 = \|\mathbf{U}_1\mathbf{y}_1\|^2 + \|\mathbf{y}_2\|^2.$$

Next, we use the expression in (33) and write the square of the norm as a function of $\mathbf{y}_2$,

$$\|\mathbf{w}\|^2 = (1 - \mathbf{1}^\top\mathbf{U}_2\mathbf{y}_2)^2\,\|\mathbf{x}\|^2 + \|\mathbf{y}_2\|^2 := f(\mathbf{y}_2).$$

To find the critical values of $f(\mathbf{y}_2)$ we calculate

$$\frac{\partial f}{\partial \mathbf{y}_2} = -2(1 - \mathbf{1}^\top\mathbf{U}_2\mathbf{y}_2)\,\|\mathbf{x}\|^2\mathbf{U}_2^\top\mathbf{1} + 2\mathbf{y}_2 = 0. \tag{38}$$

Multiplying (38) with $\mathbf{1}^\top \mathbf{U}_2$ yields

$$-\mathbf{1}^\top \mathbf{U}_2(1-\beta)\,\|\mathbf{x}\|^2 \mathbf{U}_2^\top \mathbf{1} + \beta = 0, \quad \beta := \mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2. \tag{39}$$

Solving for β we obtain

$$\mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2 = \frac{\|\mathbf{U}_2^\top \mathbf{1}\|^2 \,\|\mathbf{x}\|^2}{\|\mathbf{U}_2^\top \mathbf{1}\|^2 \,\|\mathbf{x}\|^2 + 1}. \tag{40}$$

Using this expression in (38), we derive

$$\mathbf{y}_2 = \frac{\mathbf{U}_2^\top \mathbf{1}\,\|\mathbf{x}\|^2}{\|\mathbf{U}_2^\top \mathbf{1}\|^2 \,\|\mathbf{x}\|^2 + 1}. \tag{41}$$

Observe that from (40) we have $1 - \mathbf{1}^\top \mathbf{U}_2 \mathbf{y}_2 > 0$. Thus the choice of λ_1, see (17) simplifies to the sign of γ.

Finally, substituting this $\mathbf{y}_2$ into the general solution given by (16) we arrive at (35) and (36).

The uniqueness follows from the fact that the Hessian of f,

$$\mathbf{H} = 2\mathbf{I} + 2\,\|\mathbf{x}\|^2 \mathbf{U}_2^\top \mathbf{1}\mathbf{1}^\top \mathbf{U}_2,$$

is positive definite. □

Remark 1. *It can be easily seen that* $\|\mathbf{w}_{min}\| < \|\mathbf{x}\|$, *which is the norm of* $\mathbf{w}$ *if* $y_2 \equiv 0$.

In Fig. 1 below is an illustration of the different solutions attained for the case $p = 6$ and $r = 4$. For the input data see the caption text. Note the similarity in the sets for positive and negative γ which seems to be a general property.

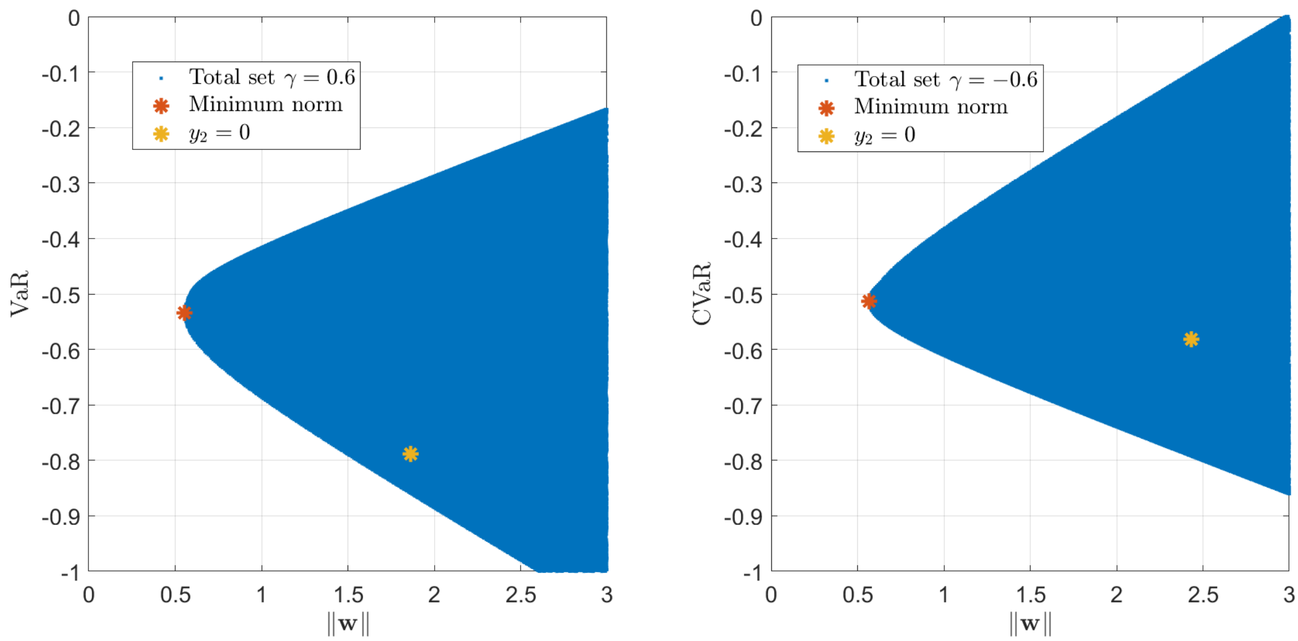

Figure 1: The solution set, in blue, when $\mathbf{y}_2 \in (-3, 3)^2$. The two stars indicate the least norm solution, red, and the solution with $\mathbf{y}_2 = 0$, yellow, for comparison. All given vectors have uniformly random elements in (0, 1) where the problem size is $p = 6$ with rank $r = 4$.

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